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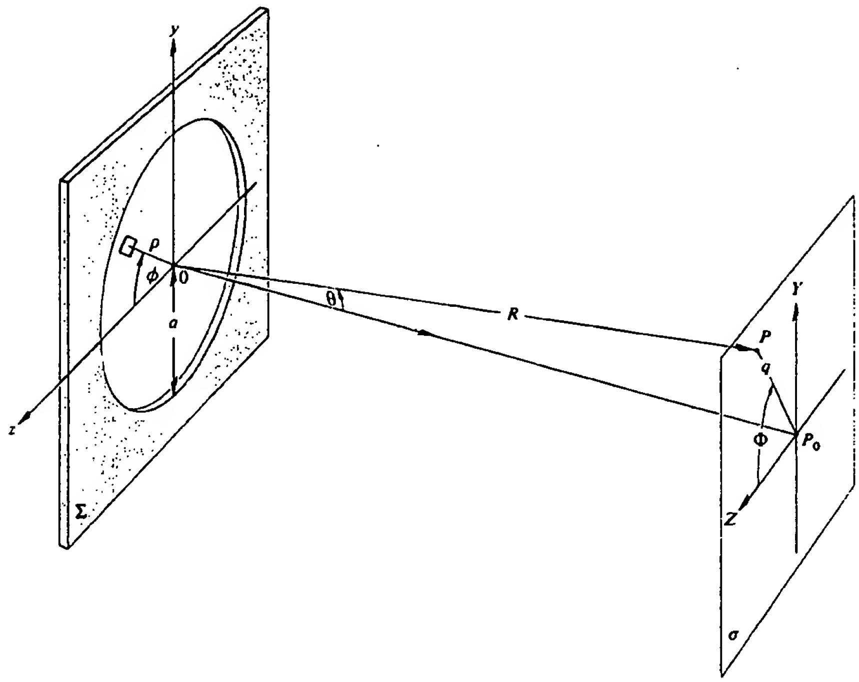


Fig. 10.26 Circular aperture geometry.

still smaller, e.g. the four corner peaks (whose coordinates correspond to appropriate combinations of $\beta' = \pm 3\pi/2$ and $\alpha' = \pm 3\pi/2$) nearest to the central maximum, each have relative irradiances of $(\frac{1}{22})^2$.

10.2.5 The Circular Aperture

Fraunhofer diffraction at a circular aperture is an effect of very great practical significance in the study of optical instrumentation. Envision a typical arrangement; plane waves impinging on a screen Σ containing a circular aperture and the consequent far-field diffraction pattern spread across a distant observing screen σ. By using a focusing lens L_2, σ can be brought in close to the aperture without changing the pattern. Now, if L_2 is positioned within and exactly fills the diffracting opening in Σ, the form of the pattern is essentially unaltered. The light wave reaching Σ is cropped so that only a circular segment propagates through L_2 to form an image in the focal plane. But quite obviously this is the same process that takes place in the eye, a telescope, microscope or camera lens. The image of a distant point source as formed by a perfectly aberration-free converging lens, is never a point, but rather some sort of diffraction pattern. We are essentially collecting only a fraction of the incident wavefront and cannot therefore hope to form a perfect image. As shown in the last section, the expression for the optical disturbance at P, arising from an arbitrary aperture in the far-field case, is

$$E = \frac{\mathcal{E}_A e^{i(\omega t - kR)}}{R} \iint_{\text{Aperture}} e^{ik(Yy + Zz)/R}\, dS. \qquad [10.41]$$

For a circular opening, symmetry would suggest introducing spherical polar coordinates in both the plane of the aperture and the plane of observation, as shown in Fig. 10.26. Therefore, let

$$z = \rho \cos\phi \qquad y = \rho \sin\phi$$

$$Z = q\cos\Phi \qquad Y = q\sin\Phi$$

and so the differential element of area is now

$$dS = \rho\, d\rho\, d\phi.$$

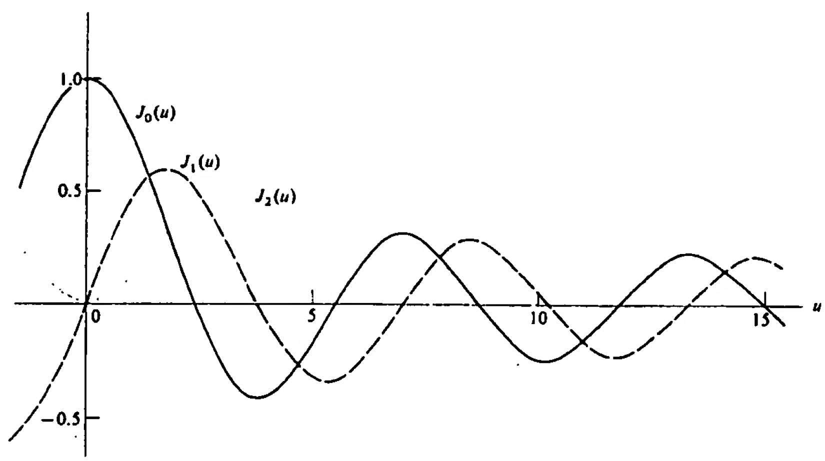


Fig. 10.27 Bessel functions.

Substituting these expressions into Eq. (10.41), it becomes

$$E = \frac{\mathcal{E}_A e^{i(\omega t - kR)}}{R} \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} e^{i(k\rho q/R)\cos(\phi - \Phi)} \rho \, d\rho \, d\phi \tag{10.46}$$

Because of the complete axial symmetry, the solution must be independent of Φ. We might just as well solve Eq. (10.46) with $\Phi = 0$ as with any other value, thereby simplifying things slightly.

The portion of the double integral associated with the variable ϕ,

$$\int_0^{2\pi} e^{i(k\rho q/R)\cos\phi} \, d\phi,$$

is one that arises quite frequently in the mathematics of physics. It is a unique function in that it cannot be reduced to any of the more common forms, such as the various hyperbolic, exponential or trigonometric functions and indeed, with the exception of these, it is perhaps the most often encountered. The quantity

$$J_0(u) = \frac{1}{2\pi} \int_0^{2\pi} e^{iu\cos v} \, dv \tag{10.47}$$

is known as the *Bessel function* (of the first kind) of order zero. More generally

$$J_m(u) = \frac{i^{-m}}{2\pi} \int_0^{2\pi} e^{i(mv + u\cos v)} \, dv \tag{10.48}$$

represents the Bessel function of order m. Numerical values of $J_0(u)$ and $J_1(u)$ are tabulated for a large range of u in most mathematical handbooks. Just like sine and cosine, the Bessel functions have series expansions and are certainly no more esoteric than these familiar childhood acquaintances. As seen in Fig. 10.27, $J_0(u)$ and $J_1(u)$ are slowly decreasing oscillatory functions that do nothing particularly dramatic.

Equation (10.46) can be rewritten as

$$E = \frac{\mathcal{E}_A e^{i(\omega t - kR)}}{R} 2\pi \int_0^a J_0(k\rho q/R)\rho \, d\rho. \tag{10.49}$$

Another general property of Bessel functions, referred to as a recurrence relation, is

$$\frac{d}{du}\left[u^m J_m(u)\right] = u^m J_{m-1}(u).$$

When $m = 1$, this clearly leads to

$$\int_0^u u' J_0(u') \, du' = u J_1(u) \tag{10.50}$$

with u' just serving as a dummy variable. If we now return to the integral in Eq. (10.49) and change the variable such that $w = k\rho q/R$, then $d\rho = (R/kq)\, dw$ and

$$\int_{\rho=0}^{\rho=a} J_0(k\rho q/R)\rho \, d\rho = (R/kq)^2 \int_{w=0}^{w=kaq/R} J_0(w) w \, dw.$$

Making use of Eq. (10.50), we get

$$E(t) = \frac{\mathcal{E}_A e^{i(\omega t - kR)}}{R} 2\pi a^2 (R/kaq) J_1(kaq/R). \qquad (10.51)$$

The irradiance at point P is $\langle(\mathrm{Re}\, E)^2\rangle$ or $\frac{1}{2}EE^*$, that is

$$I = \frac{2\mathcal{E}_A^2 A^2}{R^2}\left[\frac{J_1(kaq/R)}{kaq/R}\right]^2, \qquad (10.52)$$

where A is the area of the circular opening. To find the irradiance at the center of the pattern, i.e. at P_0, set $q = 0$. It follows from the above recurrence relation ($m = 1$) that

$$J_0(u) = \frac{d}{du}J_1(u) + \frac{J_1(u)}{u}. \qquad (10.53)$$

From Eq. (10.47) we see that $J_0(0) = 1$ and from Eq. (10.48), $J_1(0) = 0$. The ratio of $J_1(u)/u$ as u approaches zero has the same limit (L'Hôpital's rule) as the ratio of the separate derivatives of its numerator and denominator, namely $dJ_1(u)/du$ over one. But that means that the right-hand side of Eq. (10.53) is twice that limiting value, so that $J_1(u)/u = \frac{1}{2}$ at $u = 0$. The irradiance at P_0 is therefore

$$I(0) = \frac{\mathcal{E}_A^2 A^2}{2R^2}, \qquad (10.54)$$

which is the same result obtained for the rectangular opening (10.43). If R can be presumed essentially constant over the pattern, we can write

$$I = I(0)\left[\frac{2J_1(kaq/R)}{kaq/R}\right]^2. \qquad (10.55)$$

Since $\sin\theta = q/R$, the irradiance can be written as a function of θ:

$$I(\theta) = I(0)\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]^2, \qquad (10.56)$$

and as such, is plotted in Fig. 10.28. Because of the axial symmetry, the towering central maximum corresponds to a high-irradiance circular spot known as the *Airy disk*. It was Sir George Biddell Airy (1801–92) Astronomer Royal of England, who first derived Eq. (10.56). The central disk is surrounded by a dark ring which corresponds to the first zero of the function $J_1(u)$. From the standard tables $J_1(u) = 0$ when $u = 3.83$ i.e. $kaq/R = 3.83$. The radius q_1 drawn to the center of this first dark ring can be thought of as the extent of the Airy disk. It is given by

$$q_1 = 1.22\frac{R\lambda}{2a}. \qquad (10.57)$$

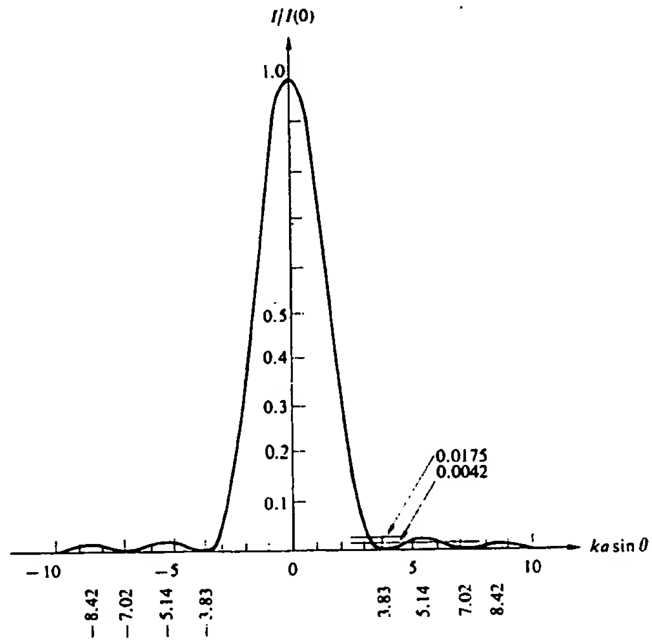


Fig. 10.28 The Airy pattern.

For a lens focused on the screen σ, the focal length $f \approx R$ and so

$$q_1 \approx 1.22\frac{f\lambda}{D}, \qquad (10.58)$$

where D is the aperture diameter i.e. $D = 2a$. (The *diameter* of the Airy disk in the visible spectrum is *very roughly* equal to the $f/\#$ of the lens in millionths of a meter.) As shown in Figs. 10.29 to 10.31, q_1 varies inversely with the hole's diameter. As D approaches λ, the Airy disk can be very large indeed and the circular aperture begins to resemble a point source of spherical waves.

The higher-order zeroes occur at values of kaq/R equal to 7.02, 10.17, etc. The secondary maxima are located where u satisfies the condition

$$\frac{d}{du}\left[\frac{J_1(u)}{u}\right] = 0,$$

which is equivalent to $J_2(u) = 0$. From the tables then, these secondary peaks occur when kaq/R equals 5.14, 8.42, 11.6, etc.; whereupon $I/I(0)$ drops from one to 0.0175, 0.0042 and 0.0016 respectively.

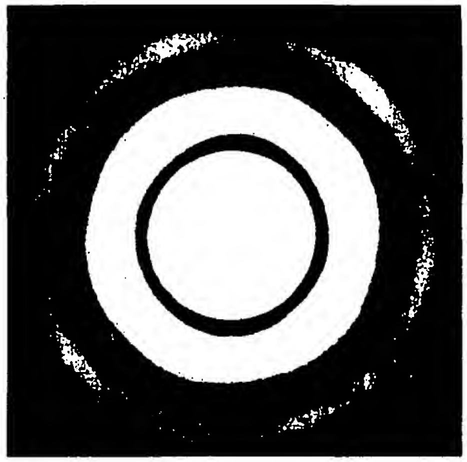


Fig. 10.29 Airy rings (0.5 mm hole diameter). [Photo by E.H.]

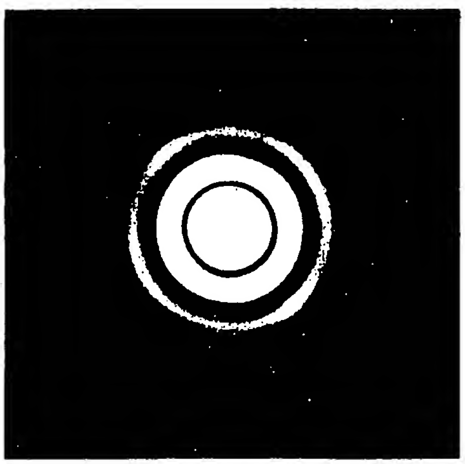
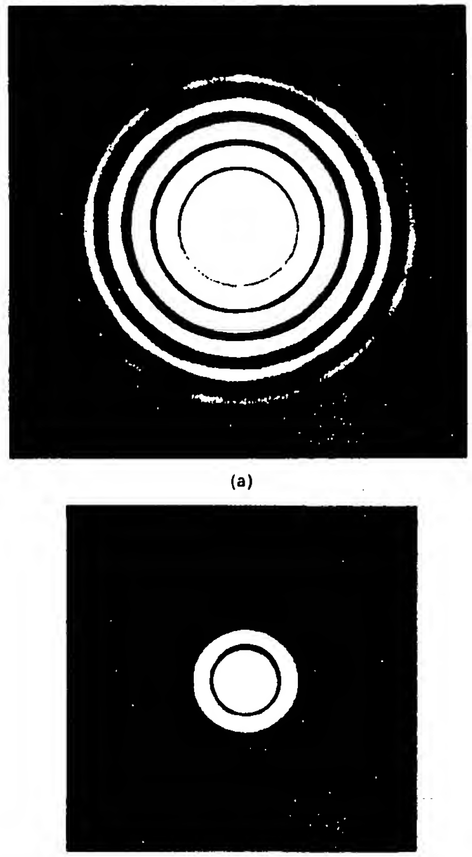


Fig. 10.30 Airy rings (1.0 mm hole diameter). [Photo by E.H.]



(a)

(b)

Fig. 10.31 (a) Airy rings—long exposure (1.5 mm hole diameter). (b) Central Airy disc—short exposure with the same aperture. [Photos by E.H.]

Circular apertures are preferable to rectangular ones as far as lens shapes go, since the circle's irradiance curve is broader around the central peak and drops off more rapidly thereafter. Exactly what fraction of the total light energy incident on σ is confined to within the various maxima is a question of interest, but one somewhat too involved to solve here.* On integrating the irradiance over a particular region of the pattern, one finds that 84% of the light arrives within the Airy disk and 91% within the bounds of the second dark ring.

* See Born and Wolf, *Principles of Optics*, p. 398 or the very fine elementary text by Towne, *Wave Phenomena*, p. 464.

10.2.6 Resolution of Imaging Systems

Imagine that we have some sort of lens system which forms an image of an extended object. If the object is self-luminous it is likely that we can regard it as made up of an array of incoherent sources. On the other hand an object seen in reflected light will surely display some phase correlation between its various scattering points. When the point sources